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Combined Motion of Three Viscous Heat-conducting Liquids in a Flat Layer

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The joint unidirectional motion of three viscous liquids under the influence of thermocapillarity forces and pressure difference has been researched. An exact stationary solution of the problem has been found. The solution of the non — stationary problem has been obtained in the form of final analytical formulas in the image using the method of Laplace transformation. By the numerical inversion of Laplace transformation the evolution of the velocity fields and of the temperature perturbation to the stationary regime for specific liquids has been obtained.

Keywords: boundary value problem, Laplace transformation, thermocapillarity.

1. Statement of the problem

Assume that there are three layers of viscous incompressible liquids with the thickness of l_1, l_2 and $l_3 - l_2$, with the interfaces $y = 0, y = l_2$, and solid walls $y = -l_1, l_3$. Motions in the layers are described by the system of viscous heat — conducting liquids equations in the absence of external forces (j = 1, 2, 3)

$$\frac{d\mathbf{u}_j}{dt} + \frac{1}{\rho_j}\nabla p_j = \nu_j \Delta \mathbf{u}_j, \ \ div\mathbf{u}_j = 0,$$
$$\frac{d\Theta_j}{dt} = \chi_j \Delta \Theta_j, \tag{1.1}$$

where $\mathbf{u}_j, p_j$ is the vector of velocity and pressure; Θ_j is the deviation from the average temperature value; ρ_j is the density; ν_j is the kinematic viscosity; χ_j — thermal diffusivity, $d/dt = \partial/\partial t + \mathbf{u}_j \cdot \nabla$. We suppose that the motion is unidirectional as

$$\mathbf{u}_j = (u_j(y,t), 0, 0).$$

Then the pressure in each liquid can be represented as $p_j = \rho_j f_j(t)x + \alpha_j(t)$ with the arbitrary f_j, α_j, and temperature — $\Theta_j = -A_j x + T_j(y,t)$ with the constants A_j. Assume that the coefficient of the surface tension σ on the interface depends on the temperature linearly: $\sigma_j(\Theta) = \sigma_j^0 - æ_j(\Theta_j - \Theta_j^0), æ_j, \sigma_j^0, \Theta_j^0 = \text{const} > 0, \ j = 1,2$. After the substitution into equations (1.1) the functions $u_j(y,t), T_j(y,t)$ satisfy the equations

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$$u_{jt} = \nu_j u_{jyy} - f_j(t), \tag{1.2}$$

$$T_{jt} = \chi_j T_{jyy} + A_j u_j. \tag{1.3}$$

The conditions of continuity of the velocities and temperatures on the interfaces (in the general view the conditions on the interface are shown in [1]) and give equalities

$$u_1(0,t) = u_2(0,t),\ u_2(l_2,t) = u_3(l_2,t), \tag{1.4}$$

$$T_1(0,t) = T_2(0,t),\ T_2(l_2,t) = T_3(l_2,t). \tag{1.5}$$

Moreover the heat fluxes are equal to

$$k_1 T_{1y}(0,t) = k_2 T_{2y}(0,t),\ k_2 T_{2y}(l_2,t) = k_3 T_{3y}(l_2,t), \tag{1.6}$$

and there are jumps of tangential stress

$$\mu_2 u_{2y}(0,t) - \mu_1 u_{1y}(0,t) = A æ_1,\ \mu_3 u_{3y}(l_2,t) - \mu_2 u_{2y}(l_2,t) = A æ_2, \tag{1.7}$$

where k_j are the heat conductivity coefficients, $\mu_j = \nu_j \rho_j$ are the dynamic viscosities. In equation (1.3) and boundary condition (1.7) $A \equiv A_1 = A_2 = A_3$ (it is a consequence of the equality of the temperature at $y = 0$ and $y = l_2$, see (1.5)). The conditions for normal stresses are reduced to pressure equality in liquids and the kinematic conditions at $y = 0,\ y = l_2$ are satisfied identically.

Since the walls $y = -l_1,\ y = l_3$ are solid, then the conditions of sticking can be written as

$$u_1(-l_1,t) = 0,\ u_3(l_3,t) = 0. \tag{1.8}$$

It is believed that the temperature gradient is constant that is

$$T_1(-l_1,t) = 0,\ T_3(l_3,t) = 0. \tag{1.9}$$

It is assumed that motion arises under the influence of thermocapillarity forces and the pressure difference from state of rest so

$$u_j(y,0) = 0, \tag{1.10}$$

$$T_j(y,0) = 0. \tag{1.11}$$

The equations (1.2)–(1.11) form two logically current tasks for the velocities u_j and the temperature perturbations T_j.

Remark 1. The considered solution of equations (1.1) is invariant relatively of to a one-parameter sub-group of continuous transformation corresponding to the operator $\partial/\partial x + \rho f(t)\partial/\partial p - A\partial/\partial\Theta$.

2. The solution of stationary problem

Suppose that velocity, pressure and temperature do not depend on time — stationary flow then the initial conditions (1.10), (1.11) are not stated. Therefore $u_j = u_j^0(y),\ T_j = T_j^0(y),\ f_j = f_j^0 = \text{const}$ and equations (1.2), (1.3) take the form $u_{jyy}^0 = f_j^0/\nu_j,\ T_{jyy}^0 = -Au_j^0/\chi_j,\ j = 1,2,3,$ it follows that

$$u_j^0 = \frac{f_j^0}{2\nu_j}y^2 + c_j^1 y + c_j^2,\ T_j^0 = -\frac{A}{\chi_j}(\frac{f_j^0}{24\nu_j}y^4 + \frac{c_j^1}{6}y^3 + \frac{c_j^2}{2}y^2) + c_j^3 y + c_j^4. \tag{2.1}$$

Constants c_j^1, c_j^2, c_j^3 и c_j^4 are determined from boundary conditions (1.4)–(1.9) and after some calculations find a representation for velocities in the dimensionless form

$$\begin{aligned}
&\bar{u}_1^0(\xi) = N(\bar{l_1}^2\xi^2 + \bar{l_1}B(\xi+1) - \bar{l_1}^2) + a_1(\xi+1), \quad -1 \leqslant \xi \leqslant 0,\\
&\bar{u}_2^0(\xi) = N(\bar{l_1}^2\bar{\mu_1}\xi^2 + \bar{l_1}B(\bar{\mu_1}\xi+1) - \bar{l_1}^2) + a_2\xi + a_1, \quad 0 \leqslant \xi \leqslant \bar{l_2}/\bar{l_1},\\
&\bar{u}_3^0(\xi) = N\bar{\mu_1}\bar{\mu_2}(\bar{l_1}^2\xi^2 + B(\bar{l_1}\xi - 1) - 1) + a_3(\xi - \frac{1}{\bar{l_1}}), \quad \bar{l_2}/\bar{l_1} \leqslant \xi \leqslant 1/\bar{l_1},
\end{aligned} \tag{2.2}$$

and temperature

$$\begin{aligned}
&\bar{T}_1^0(\xi) = N(\frac{\bar{l_1}^2}{12}\xi^4 + \frac{\bar{l_1}B}{6}\xi^3 + \frac{\bar{l_1}(B-\bar{l_1})}{2}\xi^2 + \frac{b_1}{\delta_2}(\xi+1) - \frac{\bar{l_1}B}{3} + \frac{5\bar{l_1}^2}{12}) -\\
&- \frac{a_1}{6}(\xi^3 + 3\xi^2 - 2) + \frac{b_2}{\delta_2}(\xi+1), \quad -1 \leqslant \xi \leqslant 0,\\
&\bar{T}_2^0(\xi) = N(\frac{\bar{\chi_1}\bar{\mu_1}\bar{l_1}^2}{12}\xi^4 + \frac{\bar{\chi_1}\bar{\mu_1}\bar{l_1}B}{6}\xi^3 + \frac{\bar{\chi_1}\bar{l_1}(B-\bar{l_1})}{2}\xi^2 + \frac{b_1}{\delta_2}(\bar{k_1}\xi+1) - \frac{\bar{l_1}B}{3} + \frac{5\bar{l_1}^2}{12}) -\\
&- \frac{\bar{\chi_1}}{6}(a_2\xi^3 + 3a_1\xi^2) + \frac{b_2}{\delta_2}(\bar{k_1}\xi+1) + \frac{a_1}{3}, \quad 0 \leqslant \xi \leqslant \bar{l_2}/\bar{l_1},\\
&\bar{T}_3^0(\xi) = N(\bar{\mu_1}\bar{\mu_2}\bar{\chi_2}(\frac{\bar{l_1}^2}{12}\xi^4 + \frac{\bar{l_1}B}{6}\xi^3 - \frac{(B+1)}{2}\xi^2) + b_3\xi + \frac{\bar{\mu_1}\bar{\mu_2}\bar{\chi_2}}{3\bar{l_1}^2}(\frac{5}{4} + B) - \frac{b_3}{\bar{l_1}}) -\\
&- \frac{\bar{\chi_2}a_3}{6}(\xi^3 - \frac{3\xi^2}{\bar{l_1}}) + (\frac{\bar{k_1}\bar{k_2}b_2}{\delta_2} + \frac{\bar{l_2}}{\bar{l_1}}(\frac{\bar{\chi_2}a_3}{\bar{l_1}}(\frac{\bar{l_2}}{2} - 1) - \bar{\chi_1}\bar{k_2}(a_1 + \frac{\bar{l_2}a_2}{2\bar{l_1}})))\xi \frac{\bar{k_1}\bar{k_2}b_2}{\delta_2\bar{l_1}} -\\
&- \frac{1}{\bar{l_1}^2}(\frac{\bar{\chi_2}\bar{l_2}a_3}{\bar{l_1}}(\frac{\bar{l_2}}{2} - 1) - \bar{\chi_1}\bar{k_2}\bar{l_2}(a_1 + \frac{\bar{l_2}a_2}{2\bar{l_1}}) + \frac{\bar{\chi_2}a_3}{3\bar{l_1}}),
\end{aligned} \tag{2.3}$$

where $\xi = y/l_1$, $\bar{l_1} = l_1/l_3$, $\bar{l_2} = l_2/l_3$, $\bar{\mu_1} = \mu_1/\mu_2$, $\bar{\mu_2} = \mu_2/\mu_3$, $\bar{k_1} = k_1/k_2$, $\bar{k_2} = k_2/k_3$, $\bar{\chi_1} = \chi_1/\chi_2$, $\bar{\chi_2} = \chi_1/\chi_3$, M_1, M_2 are Marangoni numbers, $N = f_1^0 l_1 l_3^2/2\nu_1^2$ is the dimensionless pressure gradient. As the characteristic velocities and temperature perturbations the relations ν_1/l_1 and $Al_1\nu_1/\chi_1$ are selected, respectively. Therefore

$$\begin{aligned}
&M_1 = \frac{A æ_1 l_1^2}{\nu_1\mu_2}, M_2 = \frac{A æ_2 l_1^2}{\nu_1\mu_2}, B = \frac{-\bar{\mu_1}\bar{\mu_2}(1 - \bar{l_2}^2) - \bar{\mu_1}\bar{l_2}^2 + \bar{l_1}^2}{\bar{\mu_1}\bar{\mu_2}(1 - \bar{l_2}) + \bar{\mu_1}\bar{l_2} + \bar{l_1}},\\
&\delta_1 = \bar{\mu_1}\bar{\mu_2}(\bar{l_2} - 1) - \bar{\mu_1}\bar{l_2} - \bar{l_1}, \delta_2 = \bar{k_1}\bar{k_2}(1 - \bar{l_2}) + \bar{k_1}\bar{l_2} + \bar{l_1},\\
&a_1 = \frac{1}{\delta_1}\Big[(\bar{l_2} + \bar{\mu_2} - \bar{\mu_2}\bar{l_2})M_1 + (1 - \bar{l_2})\bar{\mu_2}M_2)\Big], a_2 = -\frac{1}{\delta_1}\Big[\bar{l_1}M_1 + \bar{\mu_1}\bar{\mu_2}(\bar{l_2} - 1)M_2\Big],\\
&a_3 = -\frac{\bar{\mu_2}}{\delta_1}\Big[\bar{l_1}M_1 + (\bar{l_1} + \bar{\mu_1}\bar{l_2})M_2\Big],
\end{aligned}$$

$$b_1 = \frac{\bar{\mu_1}\bar{\mu_2}\bar{\chi_2}\bar{l_2}}{\bar{l_1}}(1-\bar{l_2})(\bar{l_2}(\frac{\bar{l_2}}{3}+\frac{B}{2})+B+1)+\frac{\bar{k_2}\bar{\chi_1}\bar{l_2}}{\bar{l_1}}(\bar{l_2}-1)(\bar{\mu_1}\bar{l_2}(\frac{\bar{l_2}}{3}+\frac{B}{2})+\bar{l_1}(B-\bar{l_1}))+$$
$$+\frac{\bar{\mu_1}\bar{\mu_2}\bar{\chi_2}}{3\bar{l_1}}(B+\frac{5}{4})+\frac{\bar{\mu_1}\bar{\mu_2}\bar{\chi_2}\bar{l_2}^2}{2\bar{l_1}}(\frac{\bar{l_2}}{3}(\frac{\bar{l_2}}{2}+B)-(B+1))+\frac{\bar{l_1}^2}{3}(B-\frac{5\bar{l_1}}{4})-\frac{\bar{\chi_1}\bar{l_2}^2}{2}(\frac{\bar{\mu_1}\bar{l_2}}{3\bar{l_1}}(\frac{\bar{l_2}}{2}+B)+$$
$$+(B-\frac{2}{\bar{l_1}})),$$
$$b_2 = \frac{\bar{\chi_2}\bar{l_2}a_3}{\bar{l_1}^2}(\frac{\bar{l_2}}{2}-1)(\bar{l_2}-1)-\frac{\bar{\chi_1}\bar{k_2}\bar{l_2}}{\bar{l_1}}(a_1+\frac{\bar{l_2}a_2}{2\bar{l_1}})(\bar{l_2}-1)-\frac{\bar{\chi_2}a_3}{3\bar{l_1}^2}+\frac{\bar{\chi_1}\bar{l_2}^2}{2\bar{l_1}}(a_1+\frac{\bar{l_2}a_2}{3\bar{l_1}})-$$
$$-\frac{\bar{\chi_2}\bar{l_2}^2a_3}{2\bar{l_1}^2}(\frac{\bar{l_2}}{3}-1)-\frac{a_1\bar{l_1}}{3}.$$

From the representations of solutions (2.2), (2.3) it is seen that the influence of pressure gradient and thermocapillarity forces is independent of each other. This is a consequence of the problem linearity (1.2)–(1.9).

3. The solution of the non–stationary problem using method of Laplace transformation

Apply the Laplace transformation to the problem (1.2)–(1.9). Taking into account initial conditions (1.10), (1.11), obtain in the Laplace presentation equations for velocities $\hat{U}_j(y,p)$ and the temperature perturbations $\hat{T}_j(y,p)$

$$p\hat{U}_j(y,p)=\nu_j\hat{U}_{jyy}(y,p)-F_j(p),\ p\hat{T}_j(y,p)=\chi_j\hat{T}_{jpp}(y,p)+A\hat{U}(y,p). \tag{3.1}$$

Added to (3.1) are the converted conditions (1.4)—(1.9)

$$\mu_2\hat{U}_{2y}(0,p)-\mu_1\hat{U}_{1y}(0,p)=Aæ_1/p, \tag{3.2}$$
$$\mu_3\hat{U}_{3y}(l_2,p)-\mu_2\hat{U}_{2y}(l_2,p)=Aæ_2/p, \tag{3.3}$$
$$\hat{U}_1(0,p)=\hat{U}_2(0,p),\ \hat{U}_2(l_2,p)=\hat{U}_3(l_2,p), \tag{3.4}$$
$$\hat{T}_1(0,p)=\hat{T}_2(0,p),\ \hat{T}_2(l_2,p)=\hat{T}_3(l_2,p), \tag{3.5}$$
$$\hat{U}_1(-l_1,p)=0,\ \hat{U}_3(l_3,p)=0, \tag{3.6}$$
$$\hat{T}_1(-l_1,p)=0,\ \hat{T}_3(l_3,p)=0, \tag{3.7}$$
$$k_1\hat{T}_{1y}(-l_1,p)=k_2\hat{T}_{2y}(-l_1,p),\ k_2\hat{T}_{2y}(0,p)=k_3\hat{T}_{3y}(0,p). \tag{3.8}$$

The general solution of first equation (3.1), $j=1,2,3$ of the form

$$\hat{U}_j=C_j^1\operatorname{sh}\sqrt{\frac{p}{\nu_j}}(y+l_1)+C_j^2\operatorname{ch}\sqrt{\frac{p}{\nu_j}}(y+l_1)-\frac{F_j}{p}, \tag{3.9}$$

of the second one

$$\hat{T}_j(y,p)=\hat{C}_j^1\operatorname{sh}\sqrt{\frac{p}{\chi_j}}y+\hat{C}_j^2\operatorname{ch}\sqrt{\frac{p}{\chi_j}}y+\hat{T}_{jr}, \tag{3.10}$$

where

$$\hat{T}_{jr}=\frac{A}{\sqrt{p\chi_j}}\int\limits_{\Omega_j}\hat{U}_j(z,p)\operatorname{sh}\sqrt{\frac{p}{\chi_j}}(z-y)dz$$

is the particular solution.

The constants $C_j^1, C_j^2, \hat{C}_j^1, \hat{C}_j^2$ are defined from boundary conditions (3.2)–(3.8) and one obtains ($\bar{p} = pl_1^2/\nu_1$)

$$
\begin{aligned}
&\bar{U}_1(\xi,\bar{p}) = \frac{N\bar{F}_1(\bar{p})}{\bar{p}}\left[C_1^1 \operatorname{sh}\sqrt{\bar{p}}(\xi+1) + \operatorname{ch}\sqrt{\bar{p}}(\xi+1) - 1\right] + \left[\tilde{C}_1^1 \operatorname{sh}\sqrt{\bar{p}}\xi + \tilde{C}_1^2 \operatorname{ch}\sqrt{\bar{p}}\xi\right];\\
&\bar{U}_2(\xi,\bar{p}) = \frac{N\bar{F}_1(\bar{p})}{\bar{p}}\left[C_2^1 \operatorname{sh}\sqrt{\bar{\nu}_1\bar{p}}(\xi+1) + C_2^2 \operatorname{ch}\sqrt{\bar{\nu}_1\bar{p}}(\xi+1) - \bar{\rho}_1\right] +\\
&+\left[\tilde{C}_2^1 \operatorname{sh}\sqrt{\bar{\nu}_1\bar{p}}\xi + \tilde{C}_2^2 \operatorname{ch}\sqrt{\bar{\nu}_1\bar{p}}\xi\right];\\
&\bar{U}_3(\xi,\bar{p}) = \frac{N\bar{F}_1(\bar{p})}{\bar{p}}\left[C_3^1 \operatorname{sh}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}(\xi+1) + C_3^2 \operatorname{ch}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}(\xi+1) - \bar{\rho}_2\right] +\\
&+\left[\tilde{C}_3^1 \operatorname{sh}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}\xi + \tilde{C}_3^2 \operatorname{ch}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}\xi\right],
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
a_1 &= \operatorname{sh}\sqrt{\bar{\nu}_1\bar{p}}(\bar{l}_2/\bar{l}_1) + 1 + \frac{\bar{\mu}_2}{\sqrt{\bar{\nu}_2}}\tanh\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}(1-\bar{l}_2)/\bar{l}_1 \operatorname{ch}\sqrt{\bar{\nu}_1\bar{p}}(\bar{l}_2/\bar{l}_1+1),\\
a_2 &= \operatorname{ch}\sqrt{\bar{\nu}_1\bar{p}}(\bar{l}_2/\bar{l}_1) + 1 + \frac{\bar{\mu}_2}{\sqrt{\bar{\nu}_2}}\tanh\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}(1-\bar{l}_2)/\bar{l}_1 \operatorname{sh}\sqrt{\bar{\nu}_1\bar{p}}(\bar{l}_2/\bar{l}_1+1),
\end{aligned}
$$

$$
b_1 = \frac{\bar{\mu}_1\bar{\mu}_2}{\bar{\nu}_1\bar{\nu}_2 \operatorname{ch}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}(1-\bar{l}_2)/\bar{l}_1 p} + \frac{\bar{\mu}_1}{\bar{\nu}_1} - \frac{\bar{\mu}_1\bar{\mu}_2}{\bar{\nu}_1\bar{\nu}_2},\ b_2 = 1 - \operatorname{ch}\sqrt{\bar{p}} - \frac{\bar{\mu}_1}{\bar{\nu}_1},\ b_3 = -\frac{\bar{\mu}_1}{\sqrt{\bar{\nu}_1}}\operatorname{sh}\sqrt{\bar{p}},
$$

$$
\Delta = a_2\left[\frac{\bar{\mu}_1}{\sqrt{\bar{\nu}_1}}\operatorname{ch}\sqrt{\bar{p}}\operatorname{sh}\sqrt{\bar{\nu}_1\bar{p}} - \operatorname{sh}\sqrt{\bar{p}}\operatorname{ch}\sqrt{\bar{\nu}_1\bar{p}}\right] + a_1\left[\operatorname{sh}\sqrt{\bar{p}}\operatorname{sh}\sqrt{\bar{\nu}_1\bar{p}} - \frac{\bar{\mu}_1}{\sqrt{\bar{\nu}_1}}\operatorname{ch}\sqrt{\bar{p}}\operatorname{ch}\sqrt{\bar{\nu}_1\bar{p}}\right],
$$

$$
\begin{aligned}
&C_1^1 = \frac{1}{\Delta}\left[-b_1 - (a_2b_2 + a_1b_3)\operatorname{ch}\sqrt{\bar{\nu}_1\bar{p}} + (a_2b_3 + a_1b_2)\operatorname{sh}\sqrt{\bar{\nu}_1\bar{p}}\right],\\
&C_2^1 = \frac{1}{\Delta}\left[-b_1(-\operatorname{sh}\sqrt{\bar{p}}\operatorname{sh}\sqrt{\bar{\nu}_1\bar{p}} + \frac{\bar{\mu}_1}{\sqrt{\bar{\nu}_1}}\operatorname{ch}\sqrt{\bar{p}}\operatorname{ch}\sqrt{\bar{\nu}_1\bar{p}}) + a_2(b_3\operatorname{sh}\sqrt{\bar{p}} - b_2\frac{\bar{\mu}_1}{\sqrt{\bar{\nu}_1}}\operatorname{ch}\sqrt{\bar{p}})\right],\\
&C_2^2 = \frac{1}{\Delta}\left[b_1(\frac{\bar{\mu}_1}{\sqrt{\bar{\nu}_1}}\operatorname{ch}\sqrt{\bar{p}}\operatorname{sh}\sqrt{\bar{\nu}_1\bar{p}} - \operatorname{sh}\sqrt{\bar{p}}\operatorname{ch}\sqrt{\bar{\nu}_1\bar{p}}) - a_1(b_3\operatorname{sh}\sqrt{\bar{p}} - b_2\frac{\bar{\mu}_1}{\sqrt{\bar{\nu}_1}}\operatorname{sh}\sqrt{\bar{p}})\right],\\
&C_3^1 =\\
&= \frac{\frac{\bar{\mu}_2}{\sqrt{\bar{\nu}_2}}\left[C_2^1\operatorname{ch}\sqrt{\bar{\nu}_1\bar{p}}(1+\bar{l}_2/\bar{l}_1) + C_2^2\operatorname{sh}\sqrt{\bar{\nu}_1\bar{p}}(1+\bar{l}_2/\bar{l}_1)\right]\operatorname{ch}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}(1+1/\bar{l}_1) - \bar{\rho}_2\operatorname{sh}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}(1+\bar{l}_2/\bar{l}_1)}{\operatorname{ch}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}(1-\bar{l}_2)/\bar{l}_1},\\
&C_3^2 =\\
&= \frac{\bar{\rho}_2\operatorname{sh}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}(1+\bar{l}_2/\bar{l}_1) - \frac{\bar{\mu}_2}{\sqrt{\bar{\nu}_2}}\left[C_2^1\operatorname{ch}\sqrt{\bar{\nu}_1\bar{p}}(1+\bar{l}_2/\bar{l}_1) + C_2^2\operatorname{sh}\sqrt{\bar{\nu}_1\bar{p}}(1+\bar{l}_2/\bar{l}_1)\right]\operatorname{sh}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}(1+1/\bar{l}_1))}{\operatorname{ch}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}(1-\bar{l}_2)/\bar{l}_1},
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
q &= \operatorname{sh}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}\bar{l}_2/\bar{l}_1 - \operatorname{ch}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}\bar{l}_2/\bar{l}_1\tanh\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}/\bar{l}_1,\\
q_1 &= \operatorname{ch}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}\bar{l}_2/\bar{l}_1 - \operatorname{sh}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}\bar{l}_2/\bar{l}_1\tanh\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}/\bar{l}_1,
\end{aligned}
$$

$$\tilde{C}_1^1 = \frac{1}{\bar{p}\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}}\times$$
$$\times\left[\frac{q\bar{\mu}_2 M_2 - q_1 M_1\sqrt{\bar{\nu}_2}\,\mathrm{sh}\,\sqrt{\bar{\nu}_1\bar{p}}\bar{l}_2/\bar{l}_1 + q\bar{\mu}_2 M_1\,\mathrm{ch}\,\sqrt{\bar{\nu}_1\bar{p}}\bar{l}_2/\bar{l}_1}{q_1(\frac{\bar{\mu}_1}{\sqrt{\bar{\nu}_1}}\,\mathrm{sh}\sqrt{\bar{\nu}_1\bar{p}}\bar{l}_2/\bar{l}_1+\mathrm{ch}\sqrt{\bar{\nu}_1\bar{p}}\bar{l}_2/\bar{l}_1\tanh\sqrt{\bar{p}})-\frac{q\bar{\mu}_1\bar{\mu}_2}{\sqrt{\bar{\nu}_1\bar{\nu}_2}}\,\mathrm{ch}\sqrt{\bar{\nu}_1\bar{p}}\bar{l}_2/\bar{l}_1-\frac{q\bar{\mu}_2}{\sqrt{\bar{\nu}_2}}\,\mathrm{sh}\sqrt{\bar{\nu}_1\bar{p}}\bar{l}_2/\bar{l}_1\tanh\sqrt{\bar{p}}}\right],$$
$$\tilde{C}_2^2 = \tilde{C}_1^2 = \tilde{C}_1^1\tanh\sqrt{\bar{p}},\quad \tilde{C}_2^1 = \frac{\bar{\mu}_1}{\sqrt{\bar{\nu}_1}}\tilde{C}_1^1 + \frac{M_1}{\bar{p}\sqrt{\bar{\nu}_1\bar{p}}},\quad \tilde{C}_3^2 = -\tilde{C}_3^1\tanh\sqrt{\bar{\nu}_1\bar{\nu}_2\bar{p}}/\bar{l}_1,$$
$$\tilde{C}_3^1 = \frac{\tilde{C}_1^1}{q}(\frac{\bar{\mu}_1}{\sqrt{\bar{\nu}_1}}\,\mathrm{sh}\,\sqrt{\bar{\nu}_1\bar{p}}\bar{l}_2/\bar{l}_1 + \mathrm{ch}\,\sqrt{\bar{\nu}_1\bar{p}}\bar{l}_2/\bar{l}_1\tanh\sqrt{\bar{p}}) + \frac{M_1}{q\bar{p}\sqrt{\bar{\nu}_1\bar{p}}}\,\mathrm{sh}\,\sqrt{\bar{\nu}_1\bar{p}}\bar{l}_2/\bar{l}_1.$$

Because of the complicated expressions the temperature perturbations in the Laplace representation are not given here.

Using equalities (3.10)–(3.12) and performing the calculations which are long enough one can prove limiting equalities $\lim_{p\to 0} p\tilde{\bar{T}}_j(y,p) = \bar{T}_j^0(y)$ and $\lim_{p\to 0} p\tilde{\bar{U}}_j(y,p) = \bar{u}_j^0(y)$, when $\bar{T}_j^0(y), \bar{u}_j^0(y)$ is the stationary distribution from (2.2), (2.3). Let us apply the numerical method of inversion of Laplace transformation to the obtained formulas (3.11), (3.12). The graphs only for the velocities are given because it has a real physical meanings.

Figs. 1, 2, 3 present the profiles of dimensionless velocities for the system of silicon ($\rho_1 = 956\,kg/m^3$, $\nu_1 = 10.2\cdot 10^{-6}m^2/s$, $\mu_1 = 9.71\cdot 10^{-3}kg/(m\cdot s)$, $k_1 = 0.133\ kg\cdot m/(s^3\cdot K)$, $\chi_1 = 0.0675\cdot 10^{-6}m^2/s$, $æ_1 = 6.4\cdot 10^{-5}kg/(s^2\cdot K)$) — water ($\rho_2 = 998\ kg/m^3$, $\nu_2 = 1.004\cdot 10^{-6}m^2/s$, $\mu_2 = 1.002\cdot 10^{-3}kg/(m\cdot s)$, $k_2 = 0.597\ kg\cdot m/(s^3\cdot K)$, $\chi_2 = 0.143\cdot 10^{-6}m^2/s$, $æ_2 = 15.14\cdot 10^{-5}kg/(s^2\cdot K)$) — air ($\rho_3 = 1.205\ kg/m^3$, $\nu_3 = 15.11\cdot 10^{-6}m^2/s$, $\mu_3 = 0.018\cdot 10^{-3}kg/(m\cdot s)$, $k_3 = 0.0257\ kg\cdot m/(s^3\cdot K)$, $\chi_3 = 21\cdot 10^{-6}m^2/s$) at $20°C$. It is seen that with increased of the dimensionless time $\tau = \nu_1 t/l_1^2$ the solution reaches a steady state, this being the fastest in the first and third layers. The dimensional time at $\tau = 10$ is $t = 1s$, $\bar{f}(\tau) = f_1(t)/f_1^0$ — the dimensionless pressure gradient in the first liquid. Fig. 1 illustrates the case when

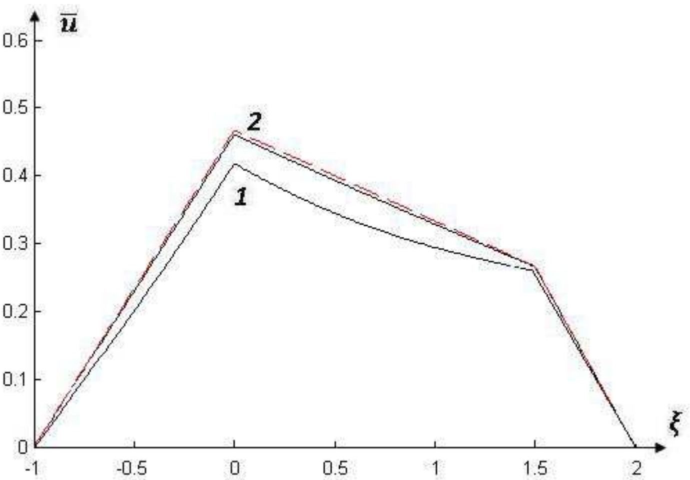

Fig. 1. Velocity profiles in the layers at $l_1 = 10^{-3}$ m; $l_2 = 1.5\cdot 10^{-3}$ m; $l_3 = 2\cdot 10^{-3}$ m; $N = 0.0001$; $M_1 = 2$; $M_2 = 3$; $\bar{f}(\tau) = 1 + e^{-\tau}cos\tau$; curve 1: $\tau = 1$; curve 2: $\tau = 3$; stationary decision (- - -)

$|N| << |a_j|$ then thermocapillarity forces are predominating and we have almost linear profiles of the velocities — the Couette flow. Fig. 2 presents the case when $|N| >> |a_j|$ then the pressure

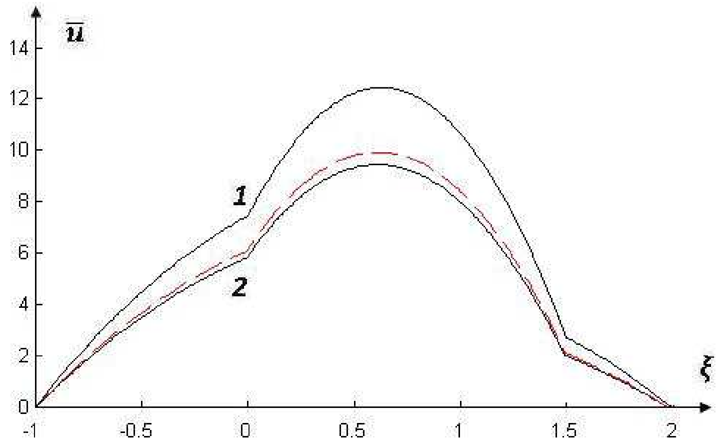

Fig. 2. Velocity profiles in the layers at $l_1 = 10^{-3}$ m; $l_2 = 1.5 \cdot 10^{-3}$ m; $l_3 = 2 \cdot 10^{-3}$ m; $N = 10$; $M_1 = 2$; $M_2 = 3$; $\bar{f}(\tau) = 1 + e^{-\tau} cos\tau$; curve 1: $\tau = 1$; curve 2: $\tau = 3$; stationary decision (- - -)

gradients in layers become the main ones and the profiles are parabolic — the Poiseuille flow. Fig. 3 shows the case of roughly equal contributions to the mechanism of the flow of the above factors. At $\bar{f}(\tau) = sin\tau$ the solution will not converge to a stationary one because the limit $\bar{f}(\tau)$ at $\tau \to \infty$ does not exist. In Fig. 4 curves 1, 2 correspond to the positive pressure gradient and curves 3, 4 to the negative one that is the motion is reversed and the process is repeated in $\tau = 2\pi$.

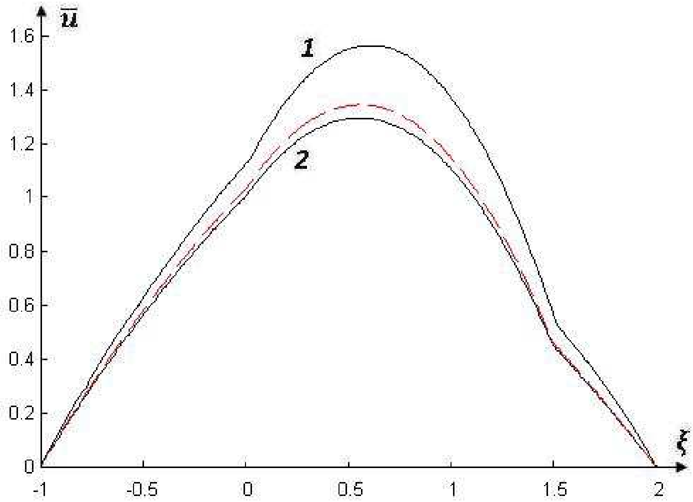

Fig. 3. Velocity profiles in the layers at $l_1 = 10^{-3}$ m; $l_2 = 1.5 \cdot 10^{-3}$ m; $l_3 = 2 \cdot 10^{-3}$ m; $N = 1$; $M_1 = 2$; $M_2 = 3$; $\bar{f}(\tau) = 1 + e^{-\tau} \cos\tau$; curve 1: $\tau = 1$; curve 2: $\tau = 3$; stationary decision (- - -)

If the dimensionless pressure gradient is

$$\bar{f}(\tau) = \begin{cases} \dfrac{\tau^*}{\tau}, & 0 \leqslant \tau \leqslant \tau^*; \\ 1 - 5e^{\tau^* - \tau}, & \tau \geqslant \tau*, \end{cases}$$

then liquids will move in the positive direction at first, in Fig. 5, curves 1,2, and at $\tau = \tau^*$ the pressure gradient changes its sign and the reverse flow occurs, curve 3. With the time increase of time the motion will reach the stationary state, curves 4, 5.

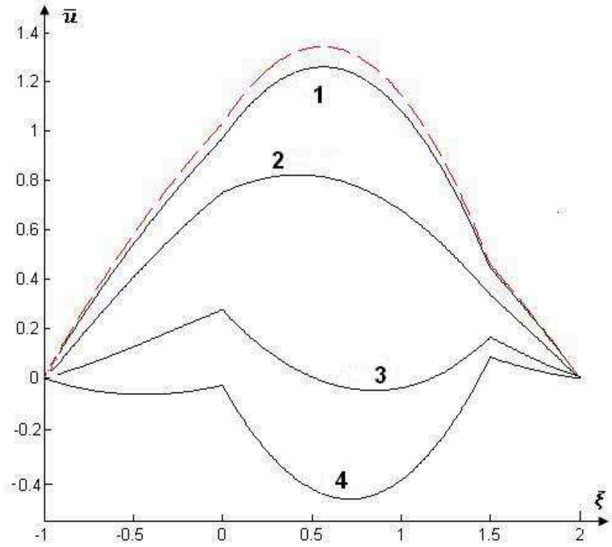

Fig. 4. Velocity profiles in the layers at $l_1 = 10^{-3}$ m; $l_2 = 1.5 \cdot 10^{-3}$ m; $l_3 = 2 \cdot 10^{-3}$ m; $N = 1$; $M_1 = 2$; $M_2 = 3$; $\bar{f}(\tau) = \sin\tau$; curve 1: $\tau = 2$; curve 2: $\tau = 3$; curve 3: $\tau = 4$; curve 4: $\tau = 5$; stationary decision (- - -)

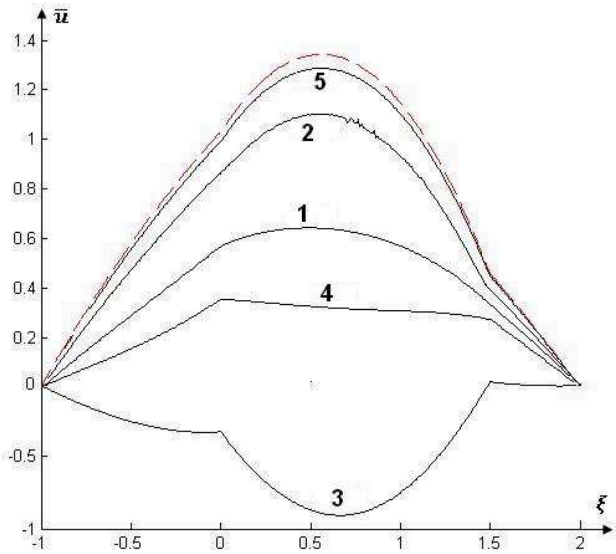

Fig. 5. Velocity profiles in the layers at $l_1 = 10^{-3}$ m; $l_2 = 1.5 \cdot 10^{-3}$ m; $l_3 = 2 \cdot 10^{-3}$ m; $N = 1$; $M_1 = 2$; $M_2 = 3$; curve 1: $\tau = 1$; curve 2: $\tau = 2$; curve 3: $\tau = 3$; curve 4: $\tau = 4$; curve 5: $\tau = 7$; $\tau^* = 2$; stationary decision (- - -)

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Комбинированное движение трёх вязких теплопроводных жидкостей в плоском слое

Елена Н. Лемешкова

Исследовано совместное однонаправленное движение трёх вязких жидкостей под действием термокапиллярных сил и перепада давления. Найдено точное стационарное решение задачи. Решение нестационарной задачи получено в виде конечных аналитических формул методом преобразования Лапласа в изображениях. Путём численного обращения преобразования Лапласа получена эволюция полей скоростей и возмущений температур к стационарному режиму для конкретных жидких сред.

Ключевые слова: краевая задача, преобразование Лапласа, термокапиллярность.